

On recovering parabolic diffusions from their time-averages

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Abstract

The paper study a possibility to recover a parabolic diffusion from its time-average when the values at the initial time are unknown. This problem can be reformulated as a new boundary value problem where a Cauchy condition is replaced by a prescribed time-average of the solution. It is shown that this new problem is well-posed. The paper establishes existence, uniqueness, and a regularity of the solution for this new problem and its modifications, including problems with singled out terminal values. MSC subject classifications: 35K20, 35Q99, 32A35.

Key words: parabolic equations, diffusion, inverse problems, ill-posed problems

1 Introduction

Parabolic diffusion equations have fundamental significance for natural and social sciences, and various boundary value problems for them were widely studied including inverse and ill-posed problems; see examples in Miller (1973), Tikhonov and Arsenin (1977), Glasko (1984), Prilepko *et al* (1984), Beck (1985), Showalter (1985), Clark and Oppenheimer (1994), Seidman (1996). According to Hadamard criterion, a boundary value problem is well-posed if there is existence and uniqueness of the solution, and if there is continuous dependence of the solution on the boundary data. Otherwise, a problem is ill-posed.

For parabolic equations, it is commonly recognized that the choice of the time for the Cauchy condition defines if a problem is well-posed or ill-posed. A classical example is the heat equation

$$u'_t(x, t) = u''_{xx}(x, t), \quad t \in [0, T].$$

The problem for this equation with the Cauchy condition $u(x, 0) \equiv \mu(x)$ at the initial time $t = 0$ is well-posed in usual classes of solutions. In contrast, the problem with the Cauchy condition $u(x, T) \equiv \mu(x)$ at the terminal time $t = T$ is ill-posed. This means that a prescribed profile of temperature at time $t = T$ cannot be achieved via an appropriate selection of the initial temperature. Respectively, the initial temperature profile cannot be recovered from the observed temperature at the terminal time. In particular, the process u is not robust with respect to small deviations of its terminal profile $u(\cdot, T)$. This makes this problem ill-posed, despite the fact that solvability and uniqueness still can be achieved for some very smooth analytical boundary data or for special selection of the domains; see e.g. Miranker (1961), Dokuchaev (2007).

Apparently there are boundary value problems that do not fit the dichotomy of the classical forward/backward well-posedness. For instance, it appears that the problems for forward heat equations are well-posed with non-local in time conditions that connects the values at different times such as

$$u(x, 0) - ku(x, T) = \mu(x) \quad \text{or} \quad u(x, 0) + \int_0^T w(t)u(x, t)dt = \mu(x),$$

for given functions μ , w , and $k \in \mathbf{R}$. Some results for parabolic equations and stochastic PDEs with these conditions replacing the Cauchy condition were obtained in Dokuchaev (2004, 2008, 2011, 2015). In these papers, $u(\cdot, 0)$ was singled out in these non-local conditions so that it counterbalanced the presence of the future values; this was achieved with restrictions on k and w .

The present paper further extends the setting with mixed in time conditions. The paper investigates solutions $u(x, t)$ of forward parabolic equations with a terminal time $T > 0$ in a domain D , with new conditions, such as

$$\int_0^T u(x, t)dt = \mu(x) \quad \text{or} \quad k_1 u(x, T) + k_2 \int_0^T u(x, t)dt = \mu(x),$$

replacing a well-posed Cauchy condition $u(x, 0) = \mu(x)$, for a given function μ and some real k_i . A crucial difference with the setting from Dokuchaev (2015) is that the present paper allows the case where the initial value $u(\cdot, 0)$ is not singled out; in this case, the initial value $u(\cdot, 0)$ is presented under the integral only, i.e. with an infinitively small weight. Moreover, the present paper allows a setting with $k_1 \neq 0$, i.e. where only the terminal value $u(\cdot, T)$ is singled out.

Formally, these new problems do not fit the framework given by the classical theory of well-posedness for parabolic equations based on the correct selection of the time for a Cauchy condition. However, we found that these new problems are well-posed for $\mu \in H^2$,

i.e. if the second partial derivatives of mu are square integrable (Theorem 1). This can be interpreted as an existence of a diffusion with a prescribed average over a time interval. Alternatively, this can be interpreted as solvability of the following inverse problem: given $\int_0^T u(x, t) dt$ for all $x \in D$, restore the entire process $u(x, t)|_{D \times [0, T]}$. It is shown below that this problem is well-posed. This is an interesting result, because it is known that, for any $c > 0$, the knowledge of values $u|_{D \times [c, T]}$ does not ensure restoring of the values $u|_{D \times [0, c]}$; this problem is ill-posed.

This result can be applied, for example, to reduce the costs of data processing for the analysis of the dynamics of heat propagation: it suffices to collect, store, and transmit only time average of temperatures rather than the entire history.

2 Problem setting

Let $D \subset \mathbf{R}^n$ be an open bounded connected domain with C^2 - smooth boundary ∂D , and let $T > 0$ be a fixed number. We consider the boundary value problems

$$\frac{\partial u}{\partial t} = Au + \varphi \quad \text{for } (x, t) \in D \times (0, T), \quad (2.1)$$

$$u(x, t) = 0 \quad \text{for } (x, t) \in \partial D \times (0, T), \quad (2.2)$$

$$\kappa u(x, T) + \int_0^T w(t)u(x, t)dt = \mu(x) \quad \text{for } x \in D. \quad (2.3)$$

Here $\kappa \in \mathbf{R}$ and a function $w(t)$ are given,

$$Au \triangleq \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sum_{j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j}(x) \right) + a_0(x, t)u(x).$$

The functions $a_{ij}(x) : D \rightarrow \mathbf{R}$ and $a_0(x) : D \rightarrow \mathbf{R}$ are continuous and bounded, and there exist continuous bounded derivatives $\partial a_{ij}(x, t)/\partial x_i$, $i, j = 1, \dots, n$. In addition, we assume that the matrix $a = \{a_{ij}\}$ is symmetric and $y^\top a(x)y \geq \delta|y|^2$ for all $x \in D$ and $y \in \mathbf{R}^n$, where $\delta > 0$ is a constant. The function $\varphi(x, t) : D \times (0, T) \rightarrow \mathbf{R}$ is measurable and square integrable. Conditions (2.1)-(2.2) describe a diffusion process in domain D .

If $\kappa \neq 0$ and $w \equiv 0$, then problem (2.1)-(2.3) is ill-posed, with a Cauchy condition $u(x, T) = \mu(x)$. To exclude this case, we assume up to the end of this paper that the following condition holds.

Condition 1 *The function μ is bounded and such that*

$$w(t) \geq 0 \quad \text{for a.e. } t \in [0, T], \quad \kappa \geq 0.$$

In addition, there exists $T_1 \in (0, T]$ such that $\text{ess inf}_{t \in [0, T_1]} w(t) > 0$.

We consider problem (2.1)-(2.3) assuming that the coefficients of A and the inputs μ and φ are known, and that the initial value $u(\cdot, 0)$ is unknown.

Some special cases

(i). If $\kappa = 0$ and $w(t) \equiv 1$, then condition (2.3) becomes

$$\int_0^T u(x, t) dt = \mu(x) \quad \text{for } x \in D. \quad (2.4)$$

Problem (2.1)-(2.2),(2.4) can be considered as a problem of recovering u from its time-average $\int_0^T u(x, t) dt$.

(ii). If $\kappa = 1$, and $w(t) \equiv \mathbb{I}_{[0, \varepsilon]}(t)$, then condition (2.3) becomes

$$u(x, T) + \int_0^\varepsilon u(x, t) dt = \mu(x) \quad \text{for } x \in D. \quad (2.5)$$

With a small $\varepsilon > 0$, solution of problem (2.1)-(2.2),(2.5) can be considered as a variation of the quasi-boundary-value method for solution of backward equation with an ill-posed condition $u(x, T) = \mu(x)$; see, e.g. Showalter (1985), Clark and Oppenheimer (1994).

Here \mathbb{I} denotes the indicator function.

Some mild restrictions will be imposed on the choice of φ for the case where $\kappa \neq 0$: it will be required that $\varphi(\cdot, t)$ features some regularity in $t \in [\theta, T]$ for some $\theta \in [0, T]$ that can be arbitrarily close to T .

Spaces and classes of functions

For a Banach space X , we denote the norm by $\|\cdot\|_X$. For a Hilbert space X , we denote the inner product by $(\cdot, \cdot)_X$.

We denote by $W_2^m(D)$ the standard Sobolev spaces of functions that belong to $L_2(D)$ together with their generalized derivatives of m th order. We denote by $\overset{0}{W}_2^1(D)$ the closure in the $W_2^1(D)$ -norm of the set of all continuously differentiable functions $u : D \rightarrow \mathbf{R}$ such that $u|_{\partial D} \equiv 0$; this is also a Hilbert space.

Let $H^0 \triangleq L_2(D)$ and $H^1 \triangleq \overset{0}{W}_2^1(D)$.

Let H^{-1} be the dual space to H^1 , with the norm $\|\cdot\|_{H^{-1}}$ such that if $u \in H^0$ then $\|u\|_{H^{-1}}$ is the supremum of $(u, v)_{H^0}$ over all $v \in H^1$ such that $\|v\|_{H^1} \leq 1$.

Let H^2 be the subspace of H^1 consisting of elements with a finite norm in $W_2^2(D)$; this is also a Hilbert space.

We denote the Lebesgue measure and the σ -algebra of Lebesgue sets in \mathbf{R}^n by $\bar{\ell}_n$ and $\bar{\mathcal{B}}_n$, respectively.

Introduce the spaces

$$\mathcal{C}_k \triangleq C([0, T]; H^k), \quad \mathcal{W}^k \triangleq L^2([0, T], \bar{\mathcal{B}}_1, \bar{\ell}_1; H^k), \quad k = -1, 0, 1, 2,$$

and the spaces

$$\mathcal{V}^k \triangleq \mathcal{W}^k \cap \mathcal{C}_{k-1}, \quad k = 1, 2,$$

with the norm $\|u\|_{\mathcal{V}} \triangleq \|u\|_{\mathcal{W}^k} + \|u\|_{\mathcal{C}_{k-1}}$.

For $\theta \in [0, T]$, we introduce a space \mathcal{U}_θ of functions $\varphi \in \mathcal{W}^0$ such that $\varphi(\cdot, t) = \varphi(\cdot, \theta) + \int_\theta^t \widehat{\varphi}(\cdot, s) ds$ for $t \in [\theta, T]$ for some $\widehat{u} \in L_1([\theta, T]; H^0)$, with the norm

$$\|\varphi\|_{\mathcal{U}_\theta} \triangleq \|\varphi\|_{\mathcal{W}^0} + \|\varphi(\cdot, \theta)\|_{H^0} + \int_\theta^T \|\widehat{\varphi}(\cdot, t)\|_{H^0} dt.$$

In particular, $\varphi(\cdot, t)$ is continuous in H^0 in $t \in [T - \theta, T]$. If $\theta = T$ then $\mathcal{U}_\theta = \mathcal{W}^0 = L_2(D \times [0, T])$.

As usual, we accept that equations (2.1)-(2.2) are satisfied for $u \in \mathcal{V}^1$ if, for any $t \in [0, T]$,

$$u(\cdot, t) = u(\cdot, 0) + \int_0^t [Au(\cdot, s) + \varphi(\cdot, s)] ds. \quad (2.6)$$

The equality here is assumed to be an equality in the space H^{-1} . Condition (2.3) is satisfied as an equality in $H^0 = L_2(D)$. The condition on ∂D is satisfied in the sense that $u(\cdot, t) \in H^1$ for a.e. t . Further, we have that $Au(\cdot, s) \in H^{-1}$ for a.e. s and the integral in (2.6) is defined as an element of H^{-1} . Hence equality (2.6) holds in the sense of equality in H^{-1} .

3 The result

Let us introduce operators $\mathcal{L} : H^k \rightarrow \mathcal{V}^{k+1}$, $k = 0, 1$, and $L : \mathcal{W}^k \rightarrow \mathcal{V}^{k+2}$, $k = -1, 0$, such that $\mathcal{L}\xi + L\varphi = v$, where v is the solution in \mathcal{V} of problem (2.1)-(2.2) with the Cauchy condition

$$u(\cdot, 0) = \xi. \quad (3.1)$$

These linear operators are continuous; see e.g. Theorems III.4.1 and IV.9.1 in Ladyzhenskaja *et al* (1968) or Theorem III.3.2 in Ladyzhenskaya (1985).

Let linear operator $M_0 : H^0 \rightarrow H^1$ be defined such that $(M_0\xi)(x) = \int_0^T w(t)u(x,t)dt + \kappa u(x,T)$, where $u = \mathcal{L}\xi \in \mathcal{V}^1$; in other words, u is the solution of problem (2.1)-(2.2) with the Cauchy condition $u(\cdot, 0) = \xi \in H^0$ and with $\varphi = 0$.

Further, let linear operator $M : \mathcal{W}^0 \rightarrow H^1$ be defined such that $(M\varphi)(x) = \int_0^T w(t)u(x,t)dt + \kappa u(x,T)$, where $u = L\varphi \in \mathcal{V}_1$; in other words, u is the solution of problem (2.1)-(2.2) with this φ and with the Cauchy condition $u(\cdot, 0) = 0$.

In these notations, $\mu = M_0u(\cdot, 0) + M\varphi$ for a solution u of problem (2.1)-(2.2).

Lemma 1 *The linear operator $M_0 : H^0 \rightarrow H^2$ is a continuous bijection; in particular, the inverse operator $M_0^{-1} : H^2 \rightarrow H^0$ is also continuous.*

Theorem 1 *Let $\theta \in [0, T]$ be such that $\theta = T$ if $\kappa = 0$ and $\theta < T$ if $\kappa \neq 0$. For any $\mu \in H^2$ and $\varphi \in \mathcal{U}_\theta$, there exists a unique solution $u \in \mathcal{V}^1$ of problem (2.1)-(2.3). Moreover, there exists $c > 0$ such that*

$$\|u\|_{\mathcal{V}^1}^2 \leq c \left(\|\mu\|_{H^2}^2 + \|\varphi\|_{\mathcal{U}_\theta}^2 \right). \quad (3.2)$$

for all $\mu \in H^2$ and $\varphi \in \mathcal{U}_\theta$. Here $c > 0$ depends only on $n, T, D, \theta, \kappa, w$, and on the coefficients of equation (2.1).

By Theorem 1, problem (2.1)-(2.3) is well-posed in the sense of Hadamard for $\mu \in H^2$ and $\varphi \in \mathcal{U}_\theta$.

The proof of Theorem 1 is based on actual construction of the solution u .

Remark 1 It can be noted that the classical results for parabolic equations imply that the operators $M_0 : H^k \rightarrow H^{k+1}$, $k = 0, 1$, and $M : \mathcal{W}^0 \rightarrow H^2$, are continuous for $\kappa = 0$, and the operators $M_0 : H^k \rightarrow H^k$, $k = 0, 1$, and $M : \mathcal{W}^0 \rightarrow H^1$, are continuous for $\kappa > 0$; see Theorems III.4.1 and IV.9.1 in Ladyzhenskaja *et al* (1968) or Theorem III.3.2 in Ladyzhenskaya (1985). The continuity of the operator $M_0 : H^0 \rightarrow H^2$ claimed in Lemma 1 requires a proof that is given in the next section.

On the properties of the solution

The solutions of new problem (2.1)-(2.3) presented in Theorem 1 have certain special features described below.

Weaker regularity than the classical problem

It appears that the solution of new problem (2.1)-(2.3) has "weaker" smoothing properties than the solution of the classical problem with standard initial Cauchy conditions. This

can be seen from the fact that problem (2.1)-(2.2),(3.1) is solvable in \mathcal{V}^2 with a initial value $u(\cdot, 0) \in H^1$ and with $\varphi \in \mathcal{W}^0$, In addition, standard problem (2.1)-(2.2),(3.1) is solvable in \mathcal{V}^1 with $u(\cdot, 0) \in H^0$ and $\varphi \in \mathcal{W}^{-1}$. On the other hand, new problem (2.1)-(2.3) with $\mu \in H^2$ provides solution in \mathcal{V}^1 only, and does not allow $\varphi \in \mathcal{W}^{-1} \setminus \mathcal{W}^0$.

Non-preserving non-negativity

For the classical problem (2.1)-(2.2),(3.1) with the standard Cauchy condition $u(x, 0) = \xi(x)$, we have that if $\xi(x) \geq 0$ and $\varphi(x, t) \geq 0$ a.e. than $u(x, t) \geq 0$ a.e. This is so-called Maximum Principle for parabolic equations; see e.g. [11], Chapter III.7).

It appears that this does not hold for condition (2.3): a solution of problem (2.1)-(2.3) with non-negative functions μ and φ is not necessarily non-negative. It follows from the Maximum Principle for parabolic equations that if $\xi(x) = u(x, 0) \geq 0$ a.e. then $\mu(x) = (M_0 \xi)(x) \geq 0$ a.e.. However, it may happen that the function $u(\cdot, 0) = M_0^{-1} \mu$ can take negative values even if $\mu(x) > 0$ in all interior points of D . This is because $\mu = M_0 u(\cdot, 0)$ actually represents a smoothing of $u(\cdot, 0)$, and this smoothing is capable of removing small negative deviations of $u(\cdot, 0)$. This feature is illustrated by a numerical example in Section 5 below.

A robustness in respect to deviation of μ in H^2

Let us discuss robustness implied by Theorem 1. Let us considered a family of functions

$$\mu_\delta(x) = \mu(x) + \delta\eta(x), \quad \varphi_\delta(x, t) = \varphi(x, t) + \delta\psi(x, t), \quad \delta > 0,$$

where $\eta \in H^2$ and $\psi \in \mathcal{U}_\theta$ represent deviations. Let u_δ be the corresponding solutions of problem (2.1)-(2.3). It follows from the linearity of the problem that

$$\|u_0 - u_\delta\|_{\mathcal{V}^1} \leq c\delta (\|\eta\|_{H^2}^2 + \|\psi\|_{\mathcal{U}_\theta}^2),$$

where $c > 0$ is the same as in (3.2); this shows that the solution is robust with respect to deviations of inputs.

However, this robustness has its limitations since the norm $\|\eta\|_{H^2}$ can be large for non-smooth or frequently oscillating η . For example, consider $\eta(x) = \eta_\theta(x) = \sin(\theta x_1) \bar{\eta}(x)$, where $\theta > 0$, $\bar{\eta} \in H^2$ is fixed and x_1 is the first component of $x = (x_1, \dots, x_n)$. In this case, $|\eta_\theta(x)| \leq |\bar{\eta}(x)|$ and $\|\eta_\theta\|_{H^2} \rightarrow +\infty$ as $\theta \rightarrow +\infty$ for a typical $\bar{\eta}$. This feature is also illustrated by a numerical example in Section 5 below.

4 Proofs

Proof of Lemma 1. It is known that there exists an orthogonal basis $\{v_k\}_{k=1}^{\infty}$ in H^0 , i.e. such that

$$(v_k, v_m)_{H^0} = 0, \quad k \neq m, \quad \|v_k\|_{H^0} = 1,$$

and such that $v_k \in H^1$ for all k , and that

$$Av_k = -\lambda_k v_k, \quad v_k|_{\partial D} = 0, \quad (4.1)$$

for some $\lambda_k \in \mathbf{R}$, $\lambda_k \rightarrow +\infty$ as $k \rightarrow +\infty$; see e.g. Ladyzhenskaya (1985), Chapter 3.4. In other words, λ_k and v_k are the eigenvalues and the corresponding eigenfunctions of the eigenvalue problem (4.1).

If $u \in \mathcal{V}^1$ is a solution of problem (2.1)-(2.3) with $\varphi = 0$, then $u(\cdot, 0) \in H^0$ is uniquely defined; it follows from the definition of \mathcal{V}^1 . Hence $\xi = u(\cdot, 0) \in H^0$ is uniquely defined. Let ξ and μ be expanded as

$$\xi = \sum_{k=1}^{\infty} \alpha_k v_k, \quad \mu = \sum_{k=1}^{\infty} \gamma_k v_k,$$

where $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\gamma_k\}_{k=1}^{\infty}$ are square-summable real sequences. By the choice of ξ , we have that $u = \mathcal{L}\xi$. Applying the Fourier method, we obtain that

$$u(x, t) = \sum_{k=1}^{\infty} \alpha_k e^{-\lambda_k t} v_k(x). \quad (4.2)$$

On the other hand,

$$\begin{aligned} \mu(x) &= \sum_{k=1}^{\infty} \gamma_k v_k(x) = \int_0^T w(t) u(x, t) dt + \kappa u(x, T) \\ &= \sum_{k=1}^{\infty} \int_0^T w(t) \alpha_k e^{-\lambda_k t} v_k(x) dt + \kappa \sum_{k=1}^{\infty} \alpha_k e^{-\lambda_k T} v_k(x) \\ &= \sum_{k=1}^{\infty} \zeta_k \alpha_k v_k(x), \end{aligned}$$

where

$$\zeta_k = \int_0^T w(t) e^{-\lambda_k t} dt + \kappa e^{-\lambda_k T}.$$

Therefore, the sequence $\{\alpha_k\}$ is uniquely defined as

$$\alpha_k = \gamma_k / \zeta_k, \quad k = 1, 2, \dots \quad (4.3)$$

Remind that we had assumed that there exists $T_1 > 0$ such that $w_* \triangleq \inf_{t \in [0, T_1]} w(t) > 0$ and that $\kappa \geq 0$. In particular, this implies that $\zeta_k > 0$ for all k . Moreover, we have that

$$\zeta_k \geq w_* \int_0^{T_1} e^{-\lambda_k t} dt + \kappa e^{-\lambda_k T} = w_* \frac{1 - e^{-\lambda_k T_1}}{\lambda_k} + \kappa e^{-\lambda_k T}.$$

In addition, we have that

$$\zeta_k \leq w_+ \int_0^{T_1} e^{-\lambda_k t} dt + \kappa e^{-\lambda_k T} = w_+ \frac{1 - e^{-\lambda_k T_1}}{\lambda_k} + \kappa e^{-\lambda_k T},$$

where $w_+ \triangleq \sup_{t \in [0, T_1]} w(t)$,

By the properties of A , we have that $\lambda_k \rightarrow +\infty$ as $k \rightarrow +\infty$, and that this sequence is non-decreasing. Hence there exists $m \geq 0$ such that $\lambda_m > 0$; respectively, $\lambda_k > 0$ for all $k \geq m$.

Let

$$\begin{aligned} c_1 &= \min \left[\zeta_1, \dots, \zeta_m, w_* \left(1 - e^{-\lambda_m T_1} \right) \right], \\ c_2 &= \max \left[\zeta_1, \dots, \zeta_m, w_+ \left(1 - e^{-\lambda_m T_1} \right) + \kappa \sup_{\lambda > 0} \lambda e^{-\lambda T} \right]. \end{aligned}$$

Clearly, $0 < c_1 < c_2$ and

$$\begin{aligned} c_1 &\leq \lambda_k \zeta_k \leq c_2, & k &\geq m, \\ c_1 &\leq \zeta_k \leq c_2, & k &< m. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} c_2^{-1} \lambda_k &\leq \zeta_k^{-1} \leq c_1^{-1} \lambda_k, & k &\geq m, \\ c_2^{-1} &\leq \zeta_k^{-1} \leq c_1^{-1}, & k &< m. \end{aligned}$$

It can be noted that estimate (4.4) is crucial for the proof; this estimate defines regularisation with T_1 is a parameter.

It follows that there exist some $C_1 > 0$ and $C_2 > 0$ such that

$$\sum_{k=1}^{\infty} \alpha_k^2 \leq C_1 \sum_{k=1}^{\infty} \gamma_k^2 \lambda_k^2 \leq C_2 \sum_{k=1}^{\infty} \alpha_k^2. \quad (4.4)$$

We have that

$$A\mu = \sum_{k=1}^{\infty} \gamma_k A v_k(x) = - \sum_{k=1}^{\infty} \gamma_k \lambda_k v_k(x)$$

and

$$\|A\mu\|_{H^0}^2 = \sum_{k=1}^{\infty} \gamma_k^2 \lambda_k^2, \quad \|\xi\|_{H^0}^2 = \sum_{k=1}^{\infty} \alpha_k^2 < +\infty. \quad (4.5)$$

Hence (4.4) can be rewritten as

$$\|\xi\|_{H^0}^2 \leq C_1 \|A\mu\|_{H^0}^2 \leq C_2 \|\xi\|_{H^0}^2. \quad (4.6)$$

Suppose that $\mu \in H^2$. In this case, $\|A\mu\|_{H^0} \leq C\|\mu\|_{H^2}$, for some $C > 0$ that is independent on μ . Thus, (4.6) implies that the operator $M_0^{-1} : H^2 \rightarrow H^0$ is continuous.

Let us prove that the operator $M_0 : H^0 \rightarrow H^2$ is continuous. From the classical estimates for parabolic equations, it follows that the operator $\mathcal{L} : H^0 \rightarrow \mathcal{W}^1$ is continuous; see, e.g., Theorem IV.9.1 in Ladyzhenskaja *et al* (1968). By the definition of the operator M_0 , it follows that the operator $M_0 : H^0 \rightarrow H^1$ is continuous.

Further, suppose that $\xi \in H^0$. Let $\mu = M_0\xi$. By (4.6), $A\mu \in H^0$. It follows that, for any $\lambda \in \mathbf{R}$, we have that $h \triangleq A\mu + \lambda\mu \in H^0$. Since the operator $M_0 : H^0 \rightarrow H^1$ is continuous, we have that $\mu \in H^1$. By the properties of the elliptic equations, it follows that there exists $\lambda \in \mathbf{R}$ and $c = c(\lambda) > 0$ such that

$$\|\mu\|_{H^2} \leq c\|h\|_{H^0} \leq c(\|A\mu\|_{H^0} + \|\lambda\mu\|_{H^0}); \quad (4.7)$$

see e.g. Theorem II.7.2 and Remark II.7.1 in Ladyzhenskaya (1975), or Theorem III.9.2 and Theorem III.10.1 in Ladyzhenskaya and Ural'ceva (1968). By (4.7), we have that

$$\|\mu\|_{H^2} \leq c_1(\|A\mu\|_{H^0} + \|\lambda\mu\|_{H^0}) \leq c_2(\|A\mu\|_{H^0} + \|\xi\|_{H^0}) \leq c_3\|\xi\|_{H^0}. \quad (4.8)$$

This completes the proof of Lemma 1.

Proof of Theorem 1. Let us show first that the operator $M : \mathcal{U}_\theta \rightarrow H^2$ is continuous. As was mentioned in Remark 1, the operator $M : \mathcal{W}^0 \rightarrow H^2$ is continuous for $\kappa = 0$; in this case, we can select $\theta = T$ and $\mathcal{U}_\theta = \mathcal{W}^0 = L_2(D \times [0, T])$.

Let us show that the operator $M : \mathcal{U}_\theta \rightarrow H^2$ is continuous for the case where $\kappa \neq 0$. By the assumptions, $\theta \neq T$ in this case and $\varphi(\cdot, t) = \varphi(\cdot, \theta) + \int_\theta^t \widehat{\varphi}(\cdot, s)ds$ for $t \in [\theta, T]$ for some $\widehat{u} \in L_1([\theta, T]; H^0)$. Without a loss of generality, let us assume that $\kappa = 1$, $\mu = M\varphi = u(x, 0)$, and $w(t) \equiv 0$; it suffices because the boundary value problem is linear.

Let v_k and λ_k be such as defined in the proof of Lemma 1.

Let μ , φ , and $\widehat{\varphi}$, be expanded as

$$\mu = \sum_{k=1}^{\infty} \gamma_k v_k, \quad \varphi(\cdot, t) = \sum_{k=1}^{\infty} \phi_k(t) v_k, \quad \widehat{\varphi}(\cdot, t) = \sum_{k=1}^{\infty} \widehat{\phi}_k(t) v_k.$$

Here $\{\gamma_k\}_{k=1}^{\infty}$ is a square-summable real sequence, the sequence $\{\phi_k(t)\}_{k=1}^{\infty} \subset L_2(0, T)$ is such that $\sum_{k=1}^{\infty} \int_0^T |\phi_k(t)|^2 dt < +\infty$, and the sequence $\{\phi_k(t)\}_{k=1}^{\infty} \subset L_1(0, T)$ is such that $\int_\theta^T \left(\sum_{k=1}^{\infty} |\widehat{\phi}_k(t)|^2 \right)^{1/2} dt < +\infty$.

Applying the Fourier method for $u = L\varphi$, we obtain that

$$u(x, T) = \sum_{k=1}^{\infty} v_k(x) \int_0^T \phi_k(t) e^{-\lambda_k(T-t)} dt. \quad (4.9)$$

On the other hand,

$$\mu(x) = \sum_{k=1}^{\infty} \gamma_k v_k(x) = u(x, T) = \sum_{k=1}^{\infty} v_k(x) \int_0^T \phi_k(t) e^{-\lambda_k(T-t)} dt = \sum_{k=1}^{\infty} v_k(x) (p_k + q_k),$$

where

$$p_k = \int_0^{\theta} \phi_k(t) e^{-\lambda_k(T-t)} dt, \quad q_k = \int_{\theta}^T \phi_k(t) e^{-\lambda_k(T-t)} dt$$

Clearly,

$$|p_k| \leq e^{-\lambda_k(T-\theta)} \int_0^{\theta} |\phi_k(t)| e^{-\lambda_k(\theta-t)} dt \leq T^{1/2} e^{-\lambda_k(T-\theta)} \|\phi_k\|_{L_2(0,T)}.$$

Further, we have that

$$\lambda_k q_k = - \int_{\theta}^T e^{-\lambda_k(T-t)} \widehat{\phi}(t) dt + \phi_k(T) - \phi_k(\theta) e^{-\lambda_k(T-\theta)}.$$

It follows that

$$\sum_{k=1}^{\infty} \lambda_k^2 p_k^2 + 2 \sum_{k=1}^{\infty} q_k^2 \leq c \|\varphi\|_{\mathcal{U}_{\theta}}^2$$

for some $c > 0$ that does not depend on φ . Hence

$$\|A\mu\|_{H^0} = \sum_{k=1}^{\infty} \lambda_k^2 \gamma_k^2 \leq 2 \sum_{k=1}^{\infty} \lambda_k^2 p_k^2 + 2 \sum_{k=1}^{\infty} \lambda_k^2 q_k^2 \leq 2c \|\varphi\|_{\mathcal{U}_{\theta}}^2.$$

Similarly to (4.7)-(4.8), we obtain that $\|\mu\|_{H^2} \leq c \|A\mu\|_{H^0}$ for some $c > 0$ that does not depend on φ . Hence the operator $M : \mathcal{U}_{\theta} \rightarrow H^2$ is continuous.

Further, it follows from the definitions of M_0 and M that

$$\mu = M_0 \xi + M\varphi.$$

Since the operator $M : \mathcal{U}_{\theta} \rightarrow H^2$ and $M_0^{-1} : H^2 \rightarrow H^0$ are continuous, it follows that $M\varphi \in H^2$ and

$$\xi = M_0^{-1}(\mu - M\varphi) \quad (4.10)$$

is uniquely defined in H^0 . Hence

$$u = \mathcal{L}\xi + L\varphi = \mathcal{L}M_0^{-1}(\mu - M\varphi) + L\varphi. \quad (4.11)$$

is an unique solution of problem (2.1)-(2.3) in \mathcal{V}^1 . By the continuity of this and other operators in (4.11), the desired estimate for u follows. This completes the proof of Theorem 1. \square

Remark 2 Equations (4.2)-(4.3) provide a numerical method for calculating $\xi = M_0^{-1}\mu$. This and (4.11) gives a numerical method for solution of problem (2.1)-(2.3).

5 Some numerical examples

An example for μ defined by (2.4)

Figure 1 shows examples of time averages $\mu(x) = \int_0^T u(x, t) dt$ and the corresponding initial profiles $u(\cdot, 0)$ restored from μ via solution of problem (2.1)-(2.2),(2.4). For these examples, we consider the problem

$$u'_t = u''_{xx} - qu, \quad u|_{\partial D} = 0, \quad \int_0^T u(x, t) dt = \mu(x).$$

for $n = 1$, $D = (0, L)$, $q \geq 0$.

To illustrate some robustness with respect to small deviations of μ , we considered a family of functions

$$\mu_{\delta, \theta}(x) = \mu(x) + \delta \eta_{\theta}(x), \quad \delta > 0, \quad \theta > 0, \quad (5.1)$$

where functions $\eta_{\theta} : D \rightarrow \mathbf{R}$ represent deviations and selected such that the norm $\|\eta_{\theta}\|_{H^2}$ is increasing in θ and that $\sup_x |\eta_{\theta}(x)|$ is bounded in θ .

For this example, we used

$$\begin{aligned} \mu(x) &= x^{1/4}(L-x)|\sin(\pi x/L)|, \\ \eta_{\theta}(x) &= x(L-x) \left(x - \frac{L}{3}\right) \left(x - \frac{2L}{3}\right) \sin(\theta x). \end{aligned} \quad (5.2)$$

With this choice, the norms $\|d^2 \eta_{\theta}(x)/dx^2\|_{H^0}$ and $\|\eta_{\theta}\|_{H^2}$ are increasing in θ .

We calculated corresponding truncated series

$$u_{\delta, \theta, N}(x, 0) = \sum_{k=1}^N \alpha_{k, \delta, \theta} v_k(x). \quad (5.3)$$

using (4.2), (4.3) with $t = 0$ and with corresponding $\alpha_k = \alpha_{k, \delta, \theta}$. Figure 1 shows these profiles for $L = 2\pi$, $T = 0.1$, $N = 50$, $\delta = 0.1$, and $\theta = 1, 3$.

It can be seen that the magnitude of deviations of $u_{\delta, \theta, N}(x, 0)$ from $u_{0, 0, N}(x, 0)$ is larger for a larger θ . As was discussed in Section 3, this is consistent with Theorem 1, because this theorem ensures robustness of the solutions with respect to deviations of μ that are small in H^2 -norm. Respectively, deviations that are small in H^0 -norm may cause large deviations of solutions.

It can be also noted that Figure 1 shows that the solution can have negative values, even given that $\mu(x) > 0$ for all $x \in D$. This illustrates the comment in Section 3 pointing out on possibility to have non-negative solution of problem (2.1)-(2.3) for nonnegative μ and φ .

We have used MATLAB; the calculation for a standard PC takes less than a second of CPU time for $N = 1000$ in the setting of Figure 1.

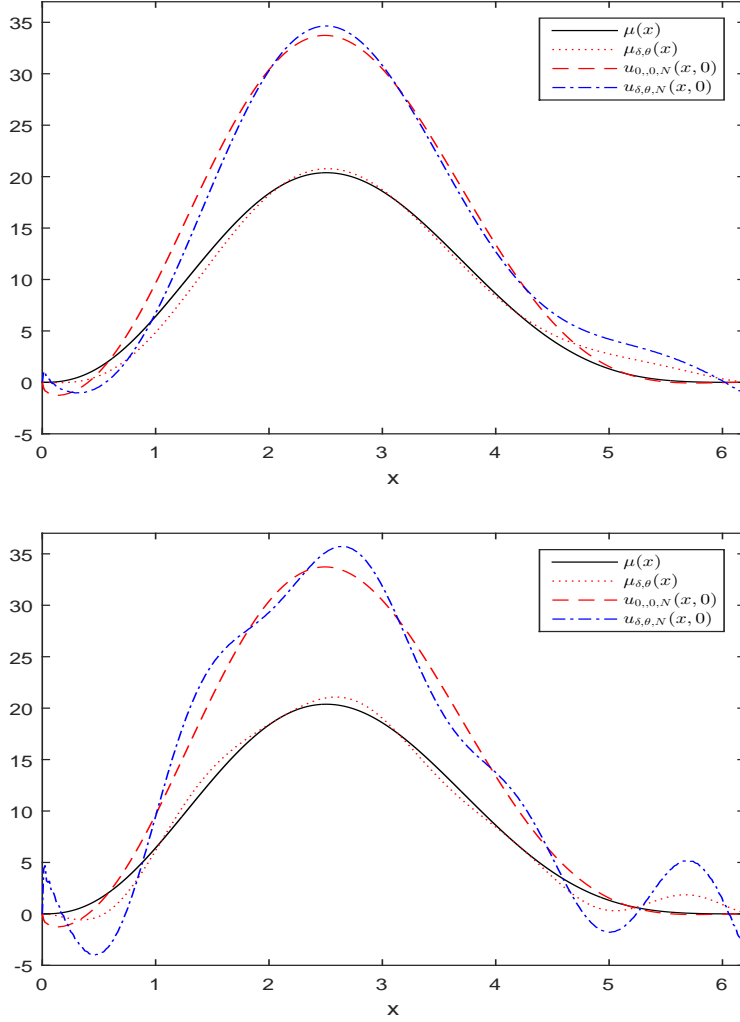


Figure 1: The profiles $\mu(x)$, $\mu_{\delta,\theta}(x)$, $u_{0,0,N}(x,0)$, and $u_{\delta,\theta,N}(x,0)$ defined by (5.1)-(5.3) with $T = 0.1$, $\delta = 0.1$, $N = 300$, $\theta = 1$ (top) and $\theta = 3$ (bottom).

An example for μ defined by (2.5) with applications to backward equations

By Theorem 1, $u(\cdot, 0)$ can be restored from observation of $\mu = \mu_\varepsilon$ for an arbitrarily small $\varepsilon > 0$, where u is a solution of problem (2.1)-(2.2),(2.5). The following example illustrates a possibility to use this for the classical problem of restoration of $u(\cdot, 0)$ from $u(\cdot, T)$. For this problem, $\mu = \mu_\varepsilon$ defined by (2.5) is actually unavailable for $\varepsilon > 0$; instead, $u(\cdot, T)$ is available. Following the approach from Showalter (1985) and Clark and Oppenheimer (1994), we presume that the integral term in (2.5) is small, and we accept $u(\cdot, T)$ as an approximation of μ_ε . This leads to acceptance of

$$u_\varepsilon(\cdot, 0) \triangleq M_{\varepsilon,0}^{-1}u(\cdot, T)$$

as an approximation of $u(\cdot, 0)$, where $M_{\varepsilon,0}$ is defined as M_0 with $\mu = \mu_\varepsilon$ defined by (2.5).

We did some numerical experiments to demonstrate potential applicability of this method. Figure 2 demonstrates the results for an example with $n = 1$, $D = (0, L)$, and with the equation $u'_t = u''_{xx} - qu$, where $q > 0$, $L > 0$. In these experiments, we first selected some profile $u(\cdot, 0)$, then calculated $u(\cdot, T)$ using the corresponding Green's function which is known for this toy forward equation; see e.g. [2], Chapter I.13. It can be noted that, for our experiment, it was sufficient to use for the Green's function truncated sin series with 50 terms. Further, for this $u(\cdot, T)$, we calculated $u_\varepsilon(\cdot, 0) \triangleq M_{\varepsilon,0}^{-1}u(\cdot, T)$ using equations (4.2)–(4.3). Finally, we compared $u_\varepsilon(\cdot, 0) \triangleq M_{\varepsilon,0}^{-1}u(\cdot, T)$ with true $u(\cdot, 0)$.

More precisely, we used truncated series

$$u_{\varepsilon,N}(x, 0) = \sum_{k=1}^N \alpha_{k,\varepsilon} v_k(x), \quad N > 0, \quad (5.4)$$

as an approximation of the solution, where $\alpha_{k,\varepsilon}$ are defined by (4.2)–(4.3) applied for $w = w_\varepsilon$.

The limit case where $\varepsilon = 0$ was not excluded; in this case,

$$u_{0,N}(x, 0) = \sum_{k=1}^N e^{\lambda_k T} g_k v_k(x) \quad (5.5)$$

is a solution based on straightforward truncation of the basis of eigenfunctions. Here $g_k \triangleq (u(\cdot, T), v_k)_{H^0}$. For comparison purpose, we calculate this solution as well.

In addition, we calculated an estimate

$$\tilde{u}_{\varepsilon,N}(x, 0) = \sum_{k=1}^N \frac{1}{\varepsilon + e^{-\lambda_k T}} g_k v_k(x). \quad (5.6)$$

This estimate is implied by the quasi-boundary-value method that suggests to replace a ill-posed boundary condition $u(x, T) = f(x)$ by a well-posed condition $\varepsilon u(x, 0) + u(x, T) = f(x)$ such as in Showalter (1985), Clark and Oppenheimer (1994).

Figure 2 shows the results for recovering $u(x, 0) = \mathbb{I}_{\{x > 1.5\}}$ using our method with $\varepsilon = 0.02$ and $N = 18$. This figure shows $u_{\varepsilon,N}(x, 0)$ (our method), $\tilde{u}_{\varepsilon,N}(x, 0)$ (quasi-boundary-value method), and $u_{0,N}(\cdot, 0)$ (straightforward truncation (5.5)). Since $\varepsilon^{-1} \int_0^\varepsilon u(x, t) dt \approx u(x, 0)$ in $L_2(D)$, it is natural to expect that the error for our solution and estimate (5.6) implied by the quasi-boundary-value method generate similar errors; Figure 2 shows that this holds for this example. In addition, it can be seen that these errors are less than the error for the estimate defined (5.5). It can be also noted that $u_{0,N}(x, 0)$ defined by (5.5) blows up for $N \geq 19$. Since analysis of the backward parabolic equations is not in the

focus of the present paper, we leave the future research the questions of selection of N and ε , convergence analysis, and more precise comparison of different methods.

We used MATLAB and a standard PC; the calculation takes less than a second of CPU time for $N = 1000$ in the setting of Figure 1, and for $N = 100$ in the setting of Figure 2.

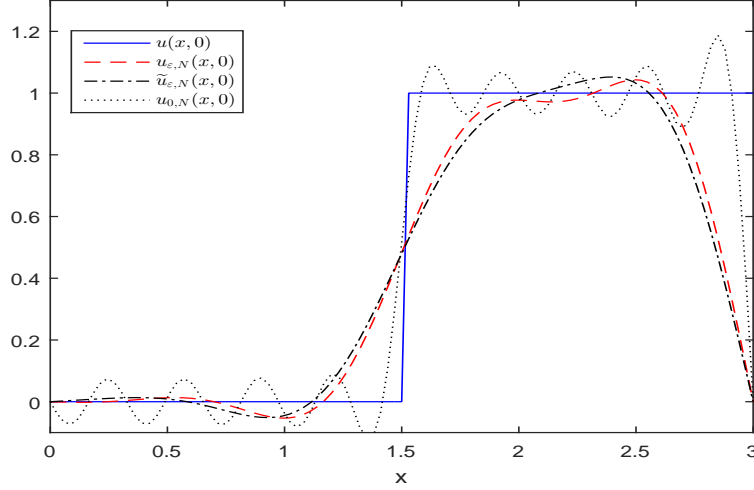


Figure 2: An initial profile $u(x, 0) = \mathbb{I}_{\{x > 1.5\}}$ and its estimates calculated for $D = (0, 3)$, $N = 18$, $T = 0.2$, and $\varepsilon = 0.05$. Here $u_{\varepsilon, N}(x, 0)$ is estimate (5.4), $\tilde{u}_{\varepsilon, N}(x, 0)$ is estimate (5.6), $u_{0, N}(x, 0)$ is estimate (5.5).

We used MATLAB and a standard PC; the calculation takes less than a second of CPU time the calculation takes less than a second of CPU time for $N = 100$ in the setting of Figure 2.

Discussion and future development

- (i). Theorem 1 can be applied, for example, to the analysis of the evolution of temperature in a domain D , with a fixed temperature on the boundary. The process $u(x, t)$ can be interpreted as the temperature at a point $x \in D$ at time t . By Theorem 1, it is possible to recover the entire evolution of the temperature in the domain if one knows the average temperature over time interval $[0, T]$.
- (ii). An analog of Theorem 1 can be obtained for the setting where problem (2.1)–(2.3) is considered for a known pair $(u(\cdot, 0), \mu)$ and for unknown φ that has to be recovered. In this case, uniqueness of recovering φ can be ensured via additional restrictions on

its dependence on time; for example, it suffices to require that $\varphi(x, t) = \psi(t)v(x)$, where ψ is a known function, and where $v \in H^0$ is unknown and has to be recovered.

- (iii). It would be interesting to extend the result on the case where the operator A is not necessarily symmetric and has coefficients depending on time. We leave this for the future research.

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